FOURIER TRANSFORMS

Definition: Let f(t) be the function defined as Integral Transform of t, then

$$\textbf{F(p)} = I(f(t) = \int_a^b f(t)k(t,p)dt$$

Where K(p, t) is kernel of integral transform of f(t) with p and t

Case 1: If k(p, t) = e^{-pt} and $a \to 0$ and $b \to \infty$, then

$$F(p) = L(f(t)) = \int_0^\infty e^{-pt} f(t)dt \text{ is called Laplace Transform of f(t)}$$

Case 2: If k(p, t) = e^{ipt} and a $a \to -\infty$ and $b \to \infty$, then

$$F(p) = F(f(t)) = \int_{-\infty}^\infty e^{ipt} f(t)dt \text{ is called infinite Fourier transform of f(t)}$$

Case 3: If k(p, t) = t^{p-1} and $a \to 0$ and $b \to \infty$, then

$$\textbf{F(p)} = M(f(t)) = \int_0^\infty t^{p-1} f(t)dt \text{ is called millens transforms of f(t)}$$

<u>FOURIER INTEGRAL THEOREM:</u>

If states if f(x) satisfies the following conditions:

(i) **F**(x) satisfies the Dirchlet conditions in every interval (-l, l)

(ii) F(x) is absolutely integrable in the interval $-\infty < x < \infty$, then

$$\textbf{f(x)} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cos\alpha(t-x)f(t)dtdp$$

<u>Different forms of Fourier Integral Formulas:</u>

By definition,

$$f(x) = \frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty \cos p(t-x)f(t)dtdp = \frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty(\cos pt\cos pxf(t) + \sin pt\sin pxf(t))dtdp$$

$$= \frac{1}{\pi}\int_0^\infty\cos(px)\int_{-\infty}^\infty f(t)\cos(pt)dtdp + \frac{1}{\pi}\int_0^\infty\sin(px)\int_{-\infty}^\infty f(t)\sin(pt)dtdp \quad \text{…. (1)}$$

When f(t) is an even function, the integral in the right hand side of (1) becomes zero. Therefore, we get

$$f(x) = \frac{2}{\pi}\int_0^\infty\cos px\int_0^\infty\cos ptf(t)dtdp.$$

This is known as the Fourier cosine integral.

When f(t) is an even function, the second integral in the right hand side of (1) becomes zero. Therefore, we get

$$f(x) = \frac{2}{\pi}\int_0^\infty\sin px\int_0^\infty\sin ptf(t)dtdp.$$

This is known as the Fourier sine integral.

Ex.1: Using Fourier integral show that $e^{-ax}-e^{-bx}=\frac{2(b^2-a^2)}{\pi}\int_0^\infty \frac{\lambda \sin \lambda x}{(\lambda^2+a^2)}d\lambda, a,b>0$

Sol. Since the integrand on R.H.S. contains sine term, we use Fourier sine integral formula.

We know that Fourier sine integral for f(x) is given by $f(x)=\frac{2}{\pi}\int_0^\infty \sin px\int_0^\infty f(t)\sin ptdtdp$

Replacing p with λ, we get $f(x)=\frac{2}{\pi}\int_0^\infty \sin \lambda x\int_0^\infty f(t)\sin \lambda tdtd\lambda$ … (1)

Here $f(x)=e^{-ax}-e^{-bx}$ or $f(t)=e^{-at}-e^{-bt}$ … (2)

Substituting (2) in (1), we get

$$f(x)=\frac{2}{\pi}\int_0^\infty \sin \lambda x\left\{\int_0^\infty (e^{-at}-e^{-bt})\sin \lambda tdt\right\}d\lambda$$

$$=\frac{2}{\pi}\int_0^\infty \sin \lambda x\left\{\int_0^\infty e^{-at}\sin \lambda tdt-\int_0^\infty e^{-bt}\sin \lambda tdt\right\}d\lambda$$

$$=\frac{2}{\pi}\int_0^\infty \sin \lambda x\left[\left\{\frac{e^{-at}}{\lambda^2+a^2}(-a\sin \lambda t-\lambda\cos \lambda t)\right\}_0^\infty-\left\{\frac{e^{-bt}}{\lambda^2+b^2}(-b\sin \lambda t-\lambda\cos \lambda t)\right\}_0^\infty\right]d\lambda$$

$$=\frac{2}{\pi}\int_0^\infty \sin \lambda x\left[\frac{\lambda}{\lambda^2+a^2}-\frac{\lambda}{\lambda^2+b^2}\right]d\lambda=\frac{2}{\pi}\int_0^\infty \sin \lambda x.\frac{\lambda(b^2-a^2)}{(\lambda^2+a^2)(\lambda^2+b^2)}d\lambda$$

$$=\frac{2(b^2-a^2)}{\pi}\int_0^\infty \frac{\lambda \sin \lambda x}{(\lambda^2+a^2)(\lambda^2+b^2)}d\lambda$$

Or $e^{-ax}-e^{-bx}=\frac{2(b^2-a^2)}{\pi}\int_0^\infty \frac{\lambda \sin \lambda x}{(\lambda^2+a^2)}d\lambda, a,b>0$

Fourier integral in complex form:

Since cos p(t – x) is an even function of p, we have

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^\infty\int_{-\infty}^\infty \cos p(t-x)f(t)dtdp....(1)$$

Also, since sin p(t – x) is an odd function of p, we have

$$0=f(x)=\frac{1}{2\pi}\int_{-\infty}^\infty\int_{-\infty}^\infty \sin p(t-x)f(t)dtdp....(2)$$

Now, multiply (2) by I and adding to (1), we get

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^\infty\int_{-\infty}^\infty [\cos p(t-x)+i\sin p(t-x)]f(t)dtdp$$

$$\Rightarrow f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{p(t-x)}f(t)\,dt\,dp$$

This is the required complex form.

Finite and Infinite Fourier Transforms and Inverse Transforms:

1. The infinite Fourier transforms of f(x):

The Fourier transform of a function f(x) is given by $F\{f(x)\}=F(p)=\int_{-\infty}^{\infty}f(x)e^{ipx}dx$

The inverse Fourier transform of F(p) is given by $f(x)=\int_{-\infty}^{\infty}F(p)e^{-ipx}dp$

2. **Fourier Sine transforms:**

i) We have $f(x)=\frac{2}{\pi}\int_0^{\infty}\sin(px)\int_0^{\infty}\sin(pt)f(t)\,dt\,dp$ which is the Fourier sine integral.

Now, write $F_s(p)=\int_0^{\infty}f(x)\sin(px)dx$. Then f(x) becomes $f(x)=\frac{2}{\pi}\int_0^{\infty}F_s(p)\sin(px)dp$

The function $F_s(p)$ is defined to be the (infinite) **Fourier Sine transform** of f(x) where $0<x<\infty$ and function f(x) is called the inverse Fourier sine transform of $F_s(p)$.

ii) The **finite Fourier Sine transform** of f(x) when $0<x<l$, is defined as

$F_s\{f(x)\}=F_s(n)=\int_0^{l}f(x)\sin\left(\frac{n\pi x}{l}\right)dx$ where n is an integer and the function f(x) is given by

$f(x)=\frac{2}{l}\sum_{n=1}^{\infty}F_s(n)\sin\left(\frac{n\pi x}{l}\right)$ is called the Inverse Fourier sine transform of $F_s(n)$.

3. **Fourier Cosine transform:**

i) We have $f(x)=\frac{2}{\pi}\int_0^{\infty}\cos(px)\int_0^{\infty}\cos(pt)f(t)\,dt\,dp$ which is the Fourier sine integral.

Now, write $F_c(p)=\int_0^{\infty}f(x)\cos(px)dx$. Then f(x) becomes $f(x)=\frac{2}{\pi}\int_0^{\infty}F_c(p)\cos(px)dp$

The function $F_c(p)$ is defined to be the (infinite) **Fourier Cosine transform** of f(x) where $0<x<\infty$ and function f(x) is called the inverse Fourier cosine transform of $F_c(p)$

ii) The **finite Fourier Cosine transform** of f(x) when $0<x<l$, is defined as

$F_c\{f(x)\}=F_c(n)=\int_0^{l}f(x)\cos\left(\frac{n\pi x}{l}\right)dx$ where n is an integer and the function f(x) is given by

$f(x)=\frac{2}{l}\sum_{n=1}^{\infty}F_c(n)\cos\left(\frac{n\pi x}{l}\right)$ is called the Inverse Fourier cosine transform of $F_c(p)$.

Properties of Fourier transform:

1. Linearity Property of Fourier transform: If F(p) and G(p) are Fourier transform of f(x) and g(x) respectively, then F[a f(x) + b g(x)] = a F(p) + b G(p). Where a and b are constants.

Proof: We have $F\{f(x)\} = F(p) = \int_{-\infty}^{\infty} f(x)e^{ipx}dx$ and $G(p) = \int_{-\infty}^{\infty} g(x)e^{ipx}dx$

$$\therefore \quad F[a\,f(x) + b\,g(x)] = \int_{-\infty}^{\infty} [a\,f(x) + b\,g(x)]\,e^{ipx}dx = \int_{-\infty}^{\infty} a\,f(x)\,e^{ipx}dx + \int_{-\infty}^{\infty} b\,g(x)\,e^{ipx}dx$$

$$= a\,F(p) + b\,G(p)$$

Note: 1. F_s[a f(x) + b g(x)] = a $F_s(p)$ + b $G_s(p)$

2. F_c[a f(x) + b g(x)] = a $F_c(p)$ + b $G_c(p)$

2. Change of scale property:

If F(p) is the complex Fourier transform of f(x) then Fourier transform of f(ax) is $\frac{1}{a}F\left(\frac{p}{a}\right)$ i.e.,

$F\{f(ax)\} = \frac{1}{a}F\left(\frac{p}{a}\right)$, a > 0.

Proof: We have $F(p) = \int_{-\infty}^{\infty} f(x)e^{ipx}dx$

$$\therefore \quad F\{f(ax)\} = \int_{-\infty}^{\infty} e^{ipx} f(x)dx$$

Put ax = t so that $dx = \frac{dt}{a}$

If $x \to -\infty$ then $t \to -\infty$ and if $x \to \infty$, then $t \to \infty$

$$\therefore \quad F\{f(ax)\} = \int_{-\infty}^{\infty} e^{ip\frac{t}{a}}.f(t)\frac{dt}{a} = \frac{1}{a}\int_{-\infty}^{\infty} e^{i\left(\frac{p}{a}\right)t}.f(t)dt = \frac{1}{a}F\left(\frac{p}{a}\right)$$

Note: 1. $F_s\{f(ax)\} = \frac{1}{a}F_s\left(\frac{p}{a}\right)$ or $F_s\left\{f\left(\frac{x}{a}\right)\right\} = aF_s(ap)$

2. $F_c\{f(ax)\} = \frac{1}{a}F_c\left(\frac{p}{a}\right)$ or $F_c\left\{f\left(\frac{x}{a}\right)\right\} = aF_c(ap)$

3. Shifting property: If F(p), is the complex Fourier transform of f(x), then the complex Fourier transform of f(x – a) is e^{ipa} F(p)

i.e., $F[f(x-a)] = e^{ipa}.F(p)$

Proof: We have $F(p) = \int_{-\infty}^{\infty} f(x)e^{ipx}dx$ …. (1)

$\therefore$ F{f(x – a)} = $\int_{-\infty}^{\infty} f(x-a)e^{ipx}dx = \int_{-\infty}^{\infty} e^{ip(t+a)} f(t)dt$ [Put x – a = t so that dx = dt. When $x \to -\infty$ then $t \to -\infty$ and if $x \to \infty$, then $t \to \infty$]

$$= e^{ipa}\int_{-\infty}^{\infty} e^{ipt} f(t)dt = e^{ipa}F(p), \text{ from (1)}$$

4. <u>Modulation Theorem</u>: If F(p) is the complex Fourier transform of f(x), then the complex Fourier transform of f(x) cos ax is $\frac{1}{2}\left[F(p+a)+F(p-a)\right]$

Proof: .We have $F(p) = \int_{-\infty}^{\infty} f(x)e^{ipx}dx$

$$\therefore \quad F\{f(x)\cos ax\} = \int_{-\infty}^{\infty} e^{ipx} f(x)\cos ax\,dx = \int_{-\infty}^{\infty} e^{ipx} f(x)\left(\frac{e^{iax}+e^{-iax}}{2}\right)dx$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} e^{i(p+a)x} f(x)dx + \int_{-\infty}^{\infty} e^{i(p-a)x} f(x)dx\right]$$

$$= \frac{1}{2}\left[F(p+a)+F(p-a)\right]$$

Note: If $F_s(p)$ and $F_c(p)$ are the Fourier sine and cosine transform of f(x) respectively. Then i) $F_S[f(x)\cos ax] = \frac{1}{2}[F_s(p+a)+F_s(p-a)]$

ii) $F_c[f(x)\sin ax] = \frac{1}{2}[F_s(p+a)-F_s(p-a)]$

iii) $F_S[f(x)\sin ax] = \frac{1}{2}[F_c(p-a)-F_c(p+a)]$

iv) $F_c[f(x)\cos ax] = \frac{1}{2}[F_c(p+a)+F_c(a-p)]$

6. $F\left\{x^n f(x)\right\} = (-i)^n \frac{d^n}{dp^n}[F(p)]$

Proof: By definition, F(p) = F{f(x)} = $\int_{-\infty}^{\infty} f(x)e^{ipx}dx$

$$\therefore \quad \frac{d}{dp}[F(p)] = \frac{d}{dp}\int_{-\infty}^{\infty} f(x)e^{ipx}dx = \int_{-\infty}^{\infty} f(x)\frac{\partial}{\partial p}(e^{ipx})dx = \int_{-\infty}^{\infty} f(x)(ix)e^{ipx}dx$$

Now $$\frac{d^2}{dp^2}[F(p)] = \frac{d}{dp}\int_{-\infty}^{\infty} f(x)(ix)e^{ipx}dx = \int_{-\infty}^{\infty} f(x)(ix)\frac{\partial}{\partial p}(e^{ipx})dx$$

$$= \int_{-\infty}^{\infty} f(x)(ix)^2 e^{ipx}dx = (i)^2\int_{-\infty}^{\infty} f(x)x^2 e^{ipx}dx$$

In general, $\frac{d^n}{dp^n}[F(p)] = (i)^n\int_{-\infty}^{\infty} x^n f(x)e^{ipx}dx = (i)^n F\{x^n(x)\}$

i.e., $F\left\{x^n f(x)\right\} = \frac{1}{(i)^n}\frac{d^n}{dp^n}[F(p)] = \frac{(i)^n}{\left((i)^n\right)^2}\frac{d^n}{dp^n}[F(p)]$

$\therefore \quad F\left\{x^n f(x)\right\} = (-i)^n \frac{d^n}{dp^n}[F(p)]$

Note: 1. $F\left\{\frac{d^n}{dx^n} f(x)\right\} = (-ip)^n F(p)$

2. $F_s\{f'(x)\} = -pF_c(p)$

Ex.1: Find the Fourier transform of f(x) defined by $f(x) = e^{-\frac{x^2}{2}}, -\infty < x < \infty$

Proof: We have F[f(x)] = $\int_{-\infty}^{\infty} f(x)e^{ipx}dx = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^{ipx}dx$

$$= \int_{-\infty}^{\infty} e^{-\frac{p^2}{2}} e^{\frac{-1}{2}(x-ip)^2} dx = e^{-\frac{p^2}{2}} \int_{-\infty}^{\infty} e^{\frac{-1}{2}(x-ip)^2} dx$$

Put $\frac{1}{\sqrt{2}}(x-ip) = t$ so that $dx = \sqrt{2}dt$

Now F[f(x)] = $e^{-\frac{p^2}{2}} \int_{-\infty}^{\infty} e^{-t^2}\sqrt{2}dt = \sqrt{2}e^{-\frac{p^2}{2}} \int_{0}^{\infty} e^{-t^2} dt = \sqrt{2}e^{-\frac{p^2}{2}}\sqrt{\pi}$

Thus Fourier transform of $e^{-\frac{x^2}{2}}$ is $e^{-\frac{p^2}{2}}$. Hence f(x) is "self – reciprocal".

Ex.2: Find the Fourier cosine and sine transform of e^{-ax}, a > 0 and hence deduce the inversion formula. (or) Deduce the integrals (i) $\int^{\infty} \frac{\cos px}{a^2+p^2}dp$ ii) $\int^{\infty} \frac{p\sin px}{a^2+p^2}dp$

Sol. Let $f(x) = e^{-ax}$. Then $F_c\{f(x)\} = \int_0^{\infty} f(x)\cos pxdx = \int_0^{\infty} e^{-ax}\cos pxdx$

$$= \left\{\frac{e^{-ax}}{a^2+p^2}[-a\cos px + p\sin px]\right\}_0^{\infty} = \frac{a}{a^2+p^2}$$

and $F_s\{f(x)\} = \int_0^{\infty} f(x)\sin pxdx = \int_0^{\infty} e^{-ax}\sin pxdx$

$$= \left\{\frac{e^{-ax}}{a^2+p^2}[-a\sin px - p\cos px]\right\}_0^{\infty} = \frac{p}{a^2+p^2}$$

i) Now by the inverse Fourier cosine transform, we get

$$f(x) = \frac{2}{\pi}\int_0^{\infty} F_c\{f(x)\}\cos pxdp = \frac{2}{\pi}\int_0^{\infty}\frac{a}{a^2+p^2}\cos pxdp$$

Since $f(x) = e^{-ax}$, therefore $e^{-ax} = \frac{2}{\pi}\int_0^{\infty}\frac{a}{a^2+p^2}\cos pxdp$

Or $\int_0^\infty \frac{\cos px}{a^2+p^2}dp = \frac{\pi}{2a}e^{-ax}$

ii) Now by the inverse Fourier sine transform, we get

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s\{f(x)\}\sin pxdp = \frac{2}{\pi}\int_0^\infty \frac{p}{a^2+p^2}\sin pxdp$$

Since $f(x) = e^{-ax}$, therefore $e^{-ax} = \frac{2}{\pi}\int_0^\infty \frac{p}{a^2+p^2}\sin pxdp$

Or $\int_0^\infty \frac{p\sin px}{a^2+p^2}dp = \frac{\pi}{2}e^{-ax}$

EX3 Using Fourier integral, show that $e^{-x}\cos x = \frac{2}{\pi}\int_0^\infty \frac{\lambda^2+2}{\lambda^4+4}\cos\lambda xd\lambda.$

<u>Fourier cosine integrals:</u>

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos px\int_0^\infty f(t)\cos ptdtdp.$$

Replacing p with λ we get

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos\lambda x\int_0^\infty f(t)\cos\lambda tdtd\lambda$$

Let $e^{-x}\cos x$

then

$$e^{-x}\cos x = \frac{2}{\pi}\int_0^\infty \cos\lambda x\int_0^\infty e^{-t}\cos t\cos\lambda tdtd\lambda$$

$$e^{-x}\cos x = \frac{1}{\pi}\int_0^\infty \cos\lambda x\int_0^\infty e^{-t}2\cos t\cos\lambda tdtd\lambda$$

$$e^{-x}\cos x = \frac{1}{\pi}\int_0^\infty \cos\lambda x\int_0^\infty e^{-t}\cos(\lambda+1)t+\cos(\lambda-1)tdtd\lambda$$

$$e^{-x}\cos x = \frac{1}{\pi}\int_0^\infty \left[\frac{1}{(\lambda+1)^2+1}+\frac{1}{(\lambda-1)^2+1}\right]\cos\lambda xd\lambda$$

$$e^{-x}\cos x = \frac{2}{\pi}\int_0^\infty \frac{(\lambda^2+2)}{(\lambda^2+2)^2-(2\lambda)^2}\cos\lambda x d\lambda$$

$$e^{-x}\cos x = \frac{2}{\pi}\int_0^\infty \frac{\lambda^2+2}{\lambda^4+4}\cos\lambda x d\lambda$$

EX 4. Show that the Fourier transforms of $e^{\frac{-x^2}{2}}$ is reciprocal

$$F\{f(x)\} = F(p) = \int_{-\infty}^{\infty} f(x)e^{ipx}dx$$

$$= \int_{-\infty}^{\infty} e^{\frac{-x^2}{2}} e^{ipx}dx$$

$$= \int_{-\infty}^{\infty} e^{\frac{-1}{2}(x-ip)^2} e^{\frac{-p^2}{2}} dx$$

$$= e^{\frac{-p^2}{2}} \int_{-\infty}^{\infty} e^{\frac{-1}{2}(x-ip)^2} dx$$

Put $\frac{1}{\sqrt{2}}(x-ip) = t \Rightarrow dx = \sqrt{2}dt$

$$F\{f(x)\} = e^{\frac{-p^2}{2}} \int_{-\infty}^{\infty} e^{-t^2}\sqrt{2}dt$$

$$= \sqrt{2}\, e^{\frac{-p^2}{2}} \int_{-\infty}^{\infty} e^{-t^2} dt$$

$$= \sqrt{2}\, e^{\frac{-p^2}{2}} \sqrt{\pi}$$

$$= \sqrt{2\pi}\, e^{\frac{-p^2}{2}}.$$

EX 5 Find Fourier sine and cosine transforms of $e^{-ax}, a > 0$ and hence deduce the integrals

(i) $\int_0^\infty \frac{\cos px}{a^2+p^2}dp.$ (ii) $\int_0^\infty \frac{p\sin px}{a^2+p^2}dp.$

Fourier cosine transforms:

Let

$$f(x) = e^{-ax}, a > 0$$

$$F_C(p) = F_C[f(x)] = \int_0^\infty f(x)\cos px\,dx$$

$$= \int_0^\infty e^{-ax}\cos px\,dx$$

$$= \left\{ \frac{e^{-ax}}{a^2 + p^2}(-a\cos px + p\sin px) \right\}_0^\infty$$

$$F_C(f(x)) = \frac{a}{a^2 + p^2}.$$

Fourier sine transforms:

Let

$$f(x) = e^{-ax}, a > 0$$

$$F_S(p) = F_S[f(x)] = \int_0^\infty f(x)\sin px\,dx$$

$$= \int_0^\infty e^{-ax}\sin px\,dx$$

$$= \left\{ \frac{e^{-ax}}{a^2 + p^2}(-a\sin px - p\cos px) \right\}_0^\infty$$

$$F_S(f(x)) = \frac{p}{a^2 + p^2}.$$

(i) Now by the inverse Fourier cosine transforms ,we get

$$f(x)=\frac{2}{\pi}\int_0^{\infty}F_c(p)\cos pxdp$$

$$e^{-ax}=\frac{2}{\pi}\int_0^{\infty}\frac{a}{a^2+p^2}\cos pxdp$$

$$e^{-ax}=\frac{2a}{\pi}\int_0^{\infty}\frac{1}{a^2+p^2}\cos pxdp$$

$$\int_0^{\infty}\frac{\cos px}{a^2+p^2}dp=\frac{\pi}{2a}e^{-ax}$$

(ii) Now by the inverse Fourier sine transforms, we get

$$f(x)=\frac{2}{\pi}\int_0^{\infty}F_S(p)\sin\ pxdp$$

$$e^{-ax}=\frac{2}{\pi}\int_0^{\infty}\frac{p}{a^2+p^2}\sin\ pxdp$$

$$\int_0^{\infty}\frac{p\sin\ px}{a^2+p^2}dp=\frac{\pi}{2}e^{-ax}$$

EX 6. Evaluate the following using parseval identity

(i) $\int_0^{\infty}\frac{x^2}{(a^2+x^2)^2}dx(a.>0)$. (ii) $\int_0^{\infty}\frac{dx}{(a^2+x^2)^2}(a.>0)$. t

Let

$$f(x)=e^{-ax}, a>0$$

$$F_C(p)=F_C[f(x)]=\int_0^{\infty}f(x)\cos pxdx \qquad F_S(p)=F_S[f(x)]=\int_0^{\infty}f(x)\sin\ pxdx$$

$$F_C(f(x))=\frac{a}{a^2+p^2}. \qquad F_S(f(x))=\frac{p}{a^2+p^2}.$$

(i) From parseval identity for Fourier transforms, we have

$$\int_0^\infty |f(x)|^2 dx = \frac{2}{\pi}\int_0^\infty |F_C(p)|^2 dp$$

$$\int_0^\infty \left|e^{-ax}\right|^2 dx = \frac{2}{\pi}\int_0^\infty \left|\frac{p}{a^2+p^2}\right|^2 dp$$

$$\int_0^\infty e^{-2ax} dx = \frac{2}{\pi}\int_0^\infty \left(\frac{p}{a^2+p^2}\right)^2 dp$$

$$\int_0^\infty \frac{p^2}{\left(a^2+p^2\right)^2} dp = \frac{\pi}{2}\left(\frac{e^{-2ax}}{-2a}\right)_0^\infty$$

$$\int_0^\infty \frac{p^2}{\left(a^2+p^2\right)^2} dp = \frac{\pi}{4a}$$

(ii) From parseval identity for Fourier transforms, we have

$$\int_0^\infty |f(x)|^2 dx = \frac{2}{\pi}\int_0^\infty |F_S(p)|^2 dp$$

$$\int_0^\infty \left|e^{-ax}\right|^2 dx = \frac{2}{\pi}\int_0^\infty \left|\frac{a}{a^2+p^2}\right|^2 dp$$

$$\int_0^\infty e^{-2ax} dx = \frac{2}{\pi}\int_0^\infty \left(\frac{a}{a^2+p^2}\right)^2 dp$$

$$\int_0^\infty \frac{a^2}{\left(a^2+p^2\right)^2} dp = \frac{\pi}{2}\left(\frac{e^{-2ax}}{-2a}\right)_0^\infty$$

$$\int_0^\infty \frac{1}{\left(a^2+p^2\right)^2} dp = \frac{\pi}{4a^3}$$